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Resonance photon generation in a vibrating cavity

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Abstract. The problem of photon creation from vacuum due to the non-stationary Casimir effect in an ideal one-dimensional Fabry–Perot cavity with vibrating walls is solved in the resonance case, when the frequency of vibrations is close to the frequency of some unperturbed electromagnetic mode: $\omega_w = p(\pi c/L_0)(1+\delta)$, $|\delta| \ll 1$, $p = 1, 2, \ldots$ (L_0 is the mean distance between the walls). An explicit analytical expression for the total energy in all the modes shows an exponential growth if $|\delta|$ is less than the dimensionless amplitude of vibrations $\varepsilon \ll 1$, the increment being proportional to $p\sqrt{\varepsilon^2 - \delta^2}$. The rate of photon generation from vacuum in the $(j+ps)$th mode goes asymptotically to a constant value $cp^2 \sin^2(\pi j/p)\sqrt{\varepsilon^2 - \delta^2}/[\pi L_0(j+ps)]$, the numbers of photons in the modes with indices $p, 2p, 3p, \ldots$ being the integrals of motion. The total number of photons in all the modes is proportional to $p^3(\varepsilon^2 - \delta^2)t^2$ in the short-time and in the long-time limits. In the case of strong detuning $|\delta| > \varepsilon$ the total energy and the total number of photons generated from vacuum oscillate with the amplitudes decreasing as $(\varepsilon/\delta)^2$ for $\varepsilon \ll |\delta|$. The special cases of $p = 1$ and $p = 2$ are studied in detail.

1. Introduction

Fifty years ago Casimir [1] showed that the presence of boundaries changes the ground state of an electromagnetic field, leading to non-trivial quantum effects like the *Casimir force* (see also [2–4]). Since then, the attention of many authors [5–40] was attracted to *non-stationary modifications* of the Casimir effect in the case of *moving boundaries* (a detailed list of publications before 1995 was given in [21]). The present paper is devoted to the special case of the *non-stationary Casimir effect* (NSCE), namely, to the effect of *photon creation from vacuum* in an ideal one-dimensional cavity (a model of the Fabry–Perot interferometer) with *vibrating* boundaries.

As was understood recently [10, 14–23], notwithstanding that the maximum velocity of the boundary achievable under laboratory conditions is very small compared with the speed of light, a gradual accumulation of the small changes in the quantum state of the field could finally result in a significant observable effect, if the boundaries of a cavity perform small oscillations at a frequency ω_w which is an integer multiple of the unperturbed eigenfrequency of the fundamental electromagnetic mode $\omega_1 = \pi c/L_0$ (where L_0 is the mean distance between the walls): $\omega_w = p\omega_1$, $p = 1, 2, \ldots$ (recall that the spectrum of the electromagnetic modes is equidistant in the case involved: the unperturbed frequency of the pth mode equals $\omega_p = p\omega_1$). The time evolution of the field in the short-time limit $\varepsilon\omega_1 t \ll 1$ (where $\varepsilon \ll 1$ is a ratio of the amplitude of vibrations to L_0) was considered in [7, 9] in

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the framework of Moore's approach [5] and in [12, 15, 28, 34, 36] in the framework of the 'instantaneous basis' method (IBM) described in section 2. The asymptotic solutions to Moore's equation in the case $\varepsilon\omega_1 t \gg 1$ were obtained in [10, 14, 27], and more general solutions were found in [16, 19, 33]. A detailed study of the problem in the framework of the IBM was given in [21] for $p = 2$ and in [22] for $p = 1$. The *short-time* limit $\varepsilon\omega_1 t \ll 1$ for an arbitrary integer value of p was considered in [28]. However, in all the cited papers the solutions were found under the condition of the *strict resonance* $\omega_w = \omega_p$ between the mechanical and electromagnetic oscillations (except the recent article [38], where a detuned three-dimensional cavity with a non-degenerate spectrum was considered). Evidently, such a condition is an idealization.

The aim of the present paper is to study the case of a *non-zero* (although small) detuning between the frequencies of the mechanical and field modes:

$$\omega_w = p\omega_1(1+\delta) \qquad |\delta| \ll 1 \tag{1.1}$$

for any integer $p = 1, 2, \ldots$, thus generalizing the results of [21, 22, 28]. It will be shown that the photons can be created from vacuum provided the dimensionless detuning parameter does not exceed the dimensionless amplitude of the wall vibrations, otherwise the total number of photons generated inside the cavity exhibits small oscillations and goes periodically to zero.

The plan of the paper is as follows. In section 2 we give general formulae related to the field quantization in a cavity with moving boundaries and derive the simplified 'reduced equations' in the resonance case. A simple explicit analytical expression for the total energy of the field for all modes is found in section 3. Section 4 is devoted to the 'semi-resonance' case $p = 1$ when the frequency of the wall is close to the fundamental frequency of the field. Under this condition new photons are not created, but the total energy of all the field modes increases exponentially with time above the threshold or oscillates in the case of a large detuning. The generic resonance case of an arbitrary $p \geqslant 2$ is analysed in section 5, and the simplifications in the case $p = 2$ are considered in section 6. A brief discussion of the results is given in section 7. Some details of calculations are given in an appendix.

2. Field quantization and reduced equations in the resonance case

Following the scheme of the field quantization in a cavity with time-dependent boundary conditions first proposed by Moore [5], we consider a cavity formed by two infinite ideal plates moving in accordance with the prescribed laws

$$x_{left}(t) = u(t) \qquad x_{right}(t) = u(t) + L(t)$$

where $L(t) > 0$ is the time dependent length of the cavity. Taking into account only the electromagnetic modes whose vector potential is directed along the z-axis ('scalar electrodynamics'), one can write down the field operator *in the Heisenberg representation* $\hat{A}(x,t)$ at $t \leqslant 0$ (when both the plates were at rest at the positions $x_{left} = 0$ and $x_{right} = L_0$) as (we assume $c = \hbar = 1$)

$$\hat{A}_{in} = 2\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \sin\frac{n\pi x}{L_0} \hat{b}_n \exp(-\mathrm{i}\omega_n t) + \text{h.c.} \tag{2.1}$$

where $\hat{b}_n$ means the usual annihilation photon operator and $\omega_n = \pi n/L_0$. The choice of coefficients in equation (2.1) corresponds to the standard form of the field Hamiltonian

$$\hat{H} \equiv \frac{1}{8\pi}\int_0^{L_0} \mathrm{d}x \left[\left(\frac{\partial A}{\partial t}\right)^2 + \left(\frac{\partial A}{\partial x}\right)^2\right] = \sum_{n=1}^{\infty} \omega_n \left(\hat{b}_n^\dagger \hat{b}_n + \tfrac{1}{2}\right). \tag{2.2}$$

For $t > 0$ the field operator can be written as

$$\hat{A}(x,t) = 2\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}\left[\hat{b}_n \psi^{(n)}(x,t) + \text{h.c.}\right].$$

To find the explicit form of functions $\psi^{(n)}(x,t)$, $n = 1, 2, \ldots$, one should take into account that the field operator must satisfy

(i) the wave equation

$$\frac{\partial^2 A}{\partial t^2} - \frac{\partial^2 A}{\partial x^2} = 0 \tag{2.3}$$

(ii) the boundary conditions

$$A(u(t),t) = A(u(t)+L(t),t) = 0 \tag{2.4}$$

(iii) the initial condition (2.1), which is equivalent to

$$\psi^{(n)}\,(x, t < 0) = \sin\frac{n\pi x}{L_0}\exp\bigl(-\mathrm{i}\omega_n t\bigr). \tag{2.5}$$

Following the approach of [12, 15, 17] we expand the function $\psi^{(n)}(x,t)$ in a series with respect to the *instantaneous basis*:

$$\psi^{(n)}(x, t > 0) = \sum_{k=1}^{\infty} Q_k^{(n)}(t)\sqrt{\frac{L_0}{L(t)}}\sin\left(\frac{\pi k[x-u(t)]}{L(t)}\right) \qquad n = 1, 2, \ldots \tag{2.6}$$

with the initial conditions

$$Q_k^{(n)}(0) = \delta_{kn} \qquad \dot{Q}_k^{(n)}(0) = -\mathrm{i}\omega_n\delta_{kn} \qquad k, n = 1, 2, \ldots\ .$$

This way we automatically satisfy both the boundary conditions (2.4) and the initial condition (2.5). Putting expression (2.6) into the wave equation (2.3), after some algebra one can arrive at an infinite set of coupled differential equations [34, 36] ($k, n = 1, 2, \ldots$)

$$\ddot{Q}_k^{(n)} + \omega_k^2(t)Q_k^{(n)} = 2\sum_{j=1}^{\infty} g_{kj}(t)\dot{Q}_j^{(n)} + \sum_{j=1}^{\infty}\dot{g}_{kj}(t)Q_j^{(n)} + \mathcal{O}(g_{kj}^2) \tag{2.7}$$

where

$$\omega_k(t) = k\pi/L(t)$$

and the time dependent antisymmetric coefficients $g_{kj}(t)$ read ($j \neq k$)

$$g_{kj} = -g_{jk} = (-1)^{k-j}\frac{2kj(\dot{L}+\dot{u}\epsilon_{kj})}{\left(j^2-k^2\right)L(t)} \qquad \epsilon_{kj} = 1-(-1)^{k-j}. \tag{2.8}$$

For $u = 0$ (the left wall at rest) the equations like (2.7), (2.8) were derived in [12, 17].

If the wall comes back to its initial position L_0 after some interval of time T, then the right-hand side of equation (2.7) disappears, so at $t > T$ one gets

$$Q_k^{(n)}(t) = \xi_k^{(n)}\mathrm{e}^{-\mathrm{i}\omega_k t} + \eta_k^{(n)}\mathrm{e}^{\mathrm{i}\omega_k t} \qquad k, n = 1, 2, \ldots \tag{2.9}$$

$\xi_k^{(n)}$ and $\eta_k^{(n)}$ being some constant coefficients. Consequently, at $t > T$ the initial annihilation operators $\hat{b}_n$ cease to be 'physical', due to the contribution of the terms with 'incorrect signs' in the exponentials $\exp(\mathrm{i}\omega_k t)$. Introducing a new set of 'physical' operators $\hat{a}_m$ and $\hat{a}_m^\dagger$, which result at $t > T$ in relations such as (2.1) and (2.2), but with $\hat{a}_m$ instead of $\hat{b}_m$, one can easily check that the two sets of operators are related by means of the Bogoliubov transformation

$$\hat{a}_m = \sum_{n=1}^{\infty}\left(\hat{b}_n\alpha_{nm} + \hat{b}_n^{\dagger}\beta_{nm}^{*}\right) \qquad m = 1, 2, \ldots \tag{2.10}$$

with the coefficients

$$\alpha_{nm} = \sqrt{\frac{m}{n}}\xi_m^{(n)} \qquad \beta_{nm} = \sqrt{\frac{m}{n}}\eta_m^{(n)}. \tag{2.11}$$

The unitarity of the transformation (2.10) implies the following constraints:

$$\sum_{m=1}^{\infty}\left(\alpha_{nm}^{*}\alpha_{km} - \beta_{nm}^{*}\beta_{km}\right) = \sum_{m=1}^{\infty}\frac{m}{n}\left(\xi_m^{(n)*}\xi_m^{(k)} - \eta_m^{(n)*}\eta_m^{(k)}\right) = \delta_{nk} \tag{2.12}$$

$$\sum_{n=1}^{\infty}\left(\alpha_{nm}^{*}\alpha_{nj} - \beta_{nm}^{*}\beta_{nj}\right) = \sum_{n=1}^{\infty}\frac{m}{n}\left(\xi_m^{(n)*}\xi_j^{(n)} - \eta_m^{(n)*}\eta_j^{(n)}\right) = \delta_{mj} \tag{2.13}$$

$$\sum_{n=1}^{\infty}\left(\beta_{nm}^{*}\alpha_{nk} - \beta_{nk}^{*}\alpha_{nm}\right) = \sum_{n=1}^{\infty}\frac{1}{n}\left(\eta_m^{(n)*}\xi_k^{(n)} - \eta_k^{(n)*}\xi_m^{(n)}\right) = 0. \tag{2.14}$$

The mean number of photons in the mth mode equals the average value of the operator $\hat{a}_m^{\dagger}\hat{a}_m$ in the initial state $|\text{in}\rangle$ (recall that we use the Heisenberg picture), since just this operator has a physical meaning at $t > T$:

$$\begin{aligned}
\mathcal{N}_m &\equiv \langle\text{in}|\hat{a}_m^{\dagger}\hat{a}_m|\text{in}\rangle \\
&= \sum_n |\beta_{nm}|^2 + \sum_{n,k}\left[\left(\alpha_{nm}^{*}\alpha_{km} + \beta_{nm}^{*}\beta_{km}\right)\langle\hat{b}_n^{\dagger}\hat{b}_k\rangle + 2\text{Re}\left(\beta_{nm}\alpha_{km}\langle\hat{b}_n\hat{b}_k\rangle\right)\right] \\
&= \sum_{n=1}^{\infty}\frac{m}{n}|\eta_m^{(n)}|^2 + \sum_{n,k=1}^{\infty}\frac{m}{\sqrt{nk}}\left(\xi_m^{(n)*}\xi_m^{(k)} + \eta_m^{(n)*}\eta_m^{(k)}\right)\langle\hat{b}_n^{\dagger}\hat{b}_k\rangle \\
&\qquad + 2\text{Re}\sum_{n,k=1}^{\infty}\frac{m}{\sqrt{nk}}\eta_m^{(n)}\xi_m^{(k)}\langle\hat{b}_n\hat{b}_k\rangle.
\end{aligned} \tag{2.15}$$

The first sum on the right-hand sides of the relations above describes the effect of the photon creation from vacuum due to the NSCE, while the other sums are different from zero only in the case of a non-vacuum initial state of the field.

To find the coefficients $\xi_k^{(n)}$ and $\eta_k^{(n)}$ one has to solve an infinite set of coupled equations (2.7) ($k = 1, 2, \ldots$) with time-dependent coefficients, moreover, each equation also contains an infinite number of terms. However, the problem can be essentially simplified, if the walls perform small oscillations at the frequency ω_w close to some unperturbed field eigenfrequency:

$$L(t) = L_0\left(1 + \varepsilon_L \sin\left[p\omega_1(1+\delta)t\right]\right) \qquad u(t) = \varepsilon_u L_0 \sin\left[p\omega_1(1+\delta)t + \varphi\right].$$

Assuming $|\varepsilon_L|, |\varepsilon_u| \sim \varepsilon \ll 1$, it is natural to look for solutions of equation (2.7) in a form similar to (2.9):

$$Q_k^{(n)}(t) = \xi_k^{(n)}\text{e}^{-\text{i}\omega_k(1+\delta)t} + \eta_k^{(n)}\text{e}^{\text{i}\omega_k(1+\delta)t} \tag{2.16}$$

but now we allow the coefficients $\xi_k^{(n)}$ and $\eta_k^{(n)}$ to be *slowly varying functions of time*. The further procedure is well known in the theory of parametrically excited systems [41–43]. First we put expression (2.16) in equation (2.7) and neglect the terms $\ddot{\xi}, \ddot{\eta}$

(bearing in mind that $\dot{\xi}, \dot{\eta} \sim \varepsilon$, while $\ddot{\xi}, \ddot{\eta} \sim \varepsilon^2$), as well as the terms proportional to $\dot{L}^2 \sim \dot{u}^2 \sim \varepsilon^2$. Multiplying the resulting equation for Q_k by the factors $\exp[\mathrm{i}\omega_k(1+\delta)t]$ and $\exp[-\mathrm{i}\omega_k(1+\delta)t]$ and performing averaging over fast oscillations with the frequencies proportional to ω_k (since the functions ξ, η practically do not change their values at the time scale of $2\pi/\omega_k$) one can verify that only the terms with the difference $j-k=\pm p$ survive on the right-hand side. Consequently, for *even* values of p the term $\dot{u}$ in $g_{kj}(t)$ does not make any contribution to the simplified equations of motion; thus only the rate of change of the cavity length $\dot{L}/L_0$ is important in this case. In contrast, if p is an *odd* number, then the field evolution depends on the velocity of the *centre of the cavity* $v_c = \dot{u} + \dot{L}/2$ and does not depend on $\dot{L}$ alone. These *interference effects* were discussed recently (in the short-time limit $\varepsilon\omega_1 t \ll 1$) in [36] (see also [23]). We assume hereafter that $u=0$ (i.e. that the left wall is at rest), since this assumption does not change anything if p is an even number, whereas one should simply replace $\dot{L}/L_0$ by $2v_c/L_0$ if p is an odd number.

The final equations for the coefficients $\xi_k^{(n)}$ and $\eta_k^{(n)}$ contain only three terms with simple *time independent* coefficients on the right-hand sides:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\xi_k^{(n)} = (-1)^p\left[(k+p)\xi_{k+p}^{(n)} - (k-p)\xi_{k-p}^{(n)}\right] + 2\mathrm{i}\gamma k\xi_k^{(n)} \tag{2.17}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\eta_k^{(n)} = (-1)^p\left[(k+p)\eta_{k+p}^{(n)} - (k-p)\eta_{k-p}^{(n)}\right] - 2\mathrm{i}\gamma k\eta_k^{(n)}. \tag{2.18}$$

The dimensionless parameters τ (a 'slow' time) and γ read ($\varepsilon \equiv \varepsilon_L$)

$$\tau = \frac{1}{2}\varepsilon\omega_1 t \qquad \gamma = \delta/\varepsilon. \tag{2.19}$$

The initial conditions are

$$\xi_k^{(n)}(0) = \delta_{kn} \qquad \eta_k^{(n)}(0) = 0. \tag{2.20}$$

Note, however, that uncoupled equations (2.17), (2.18) hold only for $k \geqslant p$. This means that they describe the evolution of *all* the Bogoliubov coefficients only if $p=1$. Then *all* the functions $\eta_k^{(n)}(t)$ are *identically equal to zero* due to the initial conditions (2.20), consequently, no photon can be created from vacuum. If $p \geqslant 2$, we have $p-1$ pair of *coupled* equations for the coefficients with lower indices $1 \leqslant k \leqslant p-1$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\xi_k^{(n)} = (-1)^p\left[(k+p)\xi_{k+p}^{(n)} - (p-k)\eta_{p-k}^{(n)}\right] + 2\mathrm{i}\gamma k\xi_k^{(n)} \tag{2.21}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\eta_k^{(n)} = (-1)^p\left[(k+p)\eta_{k+p}^{(n)} - (p-k)\xi_{p-k}^{(n)}\right] - 2\mathrm{i}\gamma k\eta_k^{(n)}. \tag{2.22}$$

In this case some functions $\eta_k^{(n)}(t)$ are not equal to zero at $t>0$, thus we have the effect of photon creation from the vacuum.

It is convenient to introduce a new set of coefficients $\rho_k^{(n)}$, whose lower indices run over all integers from $-\infty$ to ∞:

$$\rho_k^{(n)} = \begin{cases} \xi_k^{(n)} & k>0 \\ 0 & k=0 \\ -\eta_{-k}^{(n)} & k<0. \end{cases} \tag{2.23}$$

Then one can verify that equations (2.17), (2.18) and (2.21), (2.22) can be combined in a *single* set of equations ($k = \pm 1, \pm 2, \ldots$)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\rho_k^{(n)} = (-1)^p\left[(k+p)\rho_{k+p}^{(n)} - (k-p)\rho_{k-p}^{(n)}\right] + 2\mathrm{i}\gamma k\rho_k^{(n)} \tag{2.24}$$

with the initial conditions ($n = 1, 2, \ldots$)

$$\rho_k^{(n)}(0) = \delta_{kn}. \tag{2.25}$$

A remarkable feature of the set of equations (2.24) is that its solutions satisfy *exactly* the unitarity conditions (2.12)–(2.14) (although the coefficients $\xi_k^{(n)}$ and $\eta_k^{(n)}$ introduced via equation (2.16) have additional phase factors in comparison with the coefficients defined in equation (2.9), these phases do not affect the identities concerned), which can be rewritten as

$$\sum_{m=-\infty}^{\infty} m\rho_m^{(n)*}\rho_m^{(k)} = n\delta_{nk} \qquad n, k = 1, 2, \ldots \tag{2.26}$$

$$\sum_{n=1}^{\infty} \frac{m}{n}\left[\rho_m^{(n)*}\rho_j^{(n)} - \rho_{-m}^{(n)*}\rho_{-j}^{(n)}\right] = \delta_{mj} \qquad m, j = 1, 2, \ldots \tag{2.27}$$

$$\sum_{n=1}^{\infty} \frac{1}{n}\left[\rho_m^{(n)*}\rho_{-j}^{(n)} - \rho_j^{(n)*}\rho_{-m}^{(n)}\right] = 0 \qquad m, j = 1, 2, \ldots . \tag{2.28}$$

For example, calculating the derivative $I = (\mathrm{d}/\mathrm{d}\tau)\sum_{m=-\infty}^{\infty} m\rho_m^{(n)*}\rho_m^{(k)}$ with the aid of equation (2.24) and its complex conjugate counterpart one can easily verify that $I = 0$. Then the value of the right-hand side of (2.26) is a consequence of the initial conditions (2.25). The identities (2.27) and (2.28) can be verified in a similar way, if one uses instead of (2.24) the recurrence relations between the coefficients $\rho_m^{(n)}$ with the same lower index m but with different *upper* indices derived in section 5.

Due to the initial conditions (2.25) the solutions to (2.24) satisfy the relation

$$\rho_{j+mp}^{(k+np)} \equiv 0 \qquad \text{if } \ j \neq k, \quad j, k = 0, 1, \ldots, p-1$$
$$m = 0, \pm 1, \pm 2, \ldots, \quad n = 0, 1, 2, \ldots . \tag{2.29}$$

Consequently, the non-zero coefficients $\rho_m^{(n)}$ form p independent subsets

$$y_k^{(q,j)} \equiv \rho_{j+kp}^{(j+qp)} \qquad j = 0, 1, \ldots, p-1, \quad q = 0, 1, 2, \ldots, \quad k = 0, \pm 1, \pm 2, \ldots . \tag{2.30}$$

The subset $y_k^{(q,0)}$ is distinguished, because $y_k^{(q,0)} \equiv 0$ for $k \leqslant 0$ and the upper index q begins at $q = 1$. This subset is considered in detail in section 4. The generic case is studied in section 5.

3. Total energy and the rate of photon generation

It is remarkable that to calculate the total energy of the field (normalized by $\hbar\omega_1$)

$$\mathcal{E}(\tau) \equiv \sum_m m\mathcal{N}_m(\tau)$$

one does not need explicit expressions of the coefficients $\rho_m^{(n)}(\tau)$. Calculating the first and the second derivatives of $\mathcal{E}(\tau)$ with the aid of the relations (2.23)–(2.28) one can obtain a simple differential equation (see the appendix)

$$\ddot{\mathcal{E}} = 4p^2a^2\mathcal{E} + 4p^2\gamma^2\mathcal{E}(0) + \tfrac{1}{6}p^2(p^2-1) + 2p^2\gamma\sigma\,\mathrm{Im}\,(\mathcal{G}) \tag{3.1}$$

where

$$a = \sqrt{1-\gamma^2} \qquad \sigma = (-1)^p \tag{3.2}$$

$$\mathcal{G} = 2\sum_{n=1}^{\infty}\sqrt{n(n+p)}\langle \hat{b}_n^{\dagger}\hat{b}_{n+p}\rangle + \sum_{n=1}^{p-1}\sqrt{n(p-n)}\langle \hat{b}_n\hat{b}_{p-n}\rangle \tag{3.3}$$

(if $p = 1$, the last sum in (3.3) should be replaced by zero). The quantum averaging is performed over the initial state of the field (no matter pure or mixed). The initial value of the total energy is $\mathcal{E}(0) = \sum_{n=1}^{\infty} n\langle \hat{b}_n^{\dagger}\hat{b}_n\rangle$, whereas the initial value of the first derivative $\dot{\mathcal{E}}(\tau)$ reads (see the appendix)

$$\dot{\mathcal{E}}(0) = -p\sigma\,\mathrm{Re}\,(\mathcal{G}). \tag{3.4}$$

Consequently, the solution to equation (3.1) can be expressed as

$$\mathcal{E}(\tau) = \mathcal{E}(0) + \frac{2\sinh^2(pa\tau)}{a^2}\left[\mathcal{E}(0) + \frac{p^2-1}{24} + \frac{\gamma\sigma}{2}\mathrm{Im}\,(\mathcal{G})\right] - \sigma\,\mathrm{Re}\,(\mathcal{G})\frac{\sinh(2pa\tau)}{2a}. \tag{3.5}$$

We see that the total energy increases exponentially at $\tau \to \infty$, provided $\gamma < 1$. In the special case $\gamma = 0$ such asymptotic behaviour of the total energy was obtained also in the frameworks of other approaches in [16, 17, 19, 20]. Here we have found the explicit dependence of the total energy on time in the whole interval $0 \leqslant \tau < \infty$, as well as a non-trivial dependence on the initial state of field, which is contained in the constant parameter $\mathcal{G}$. This parameter is equal to zero for initial Fock or thermal states of the field. However, in the generic case $\mathcal{G}$ is different from zero, and it can affect significantly the total energy, if $\mathcal{E}(0) \gg 1$. Consider, for example, the case $p = 2$. If initially the first mode ($n = 1$) was in the coherent state $|\alpha\rangle$ with $\alpha = |\alpha|\mathrm{e}^{\mathrm{i}\phi}$, $|\alpha| \gg 1$, and all other modes were not excited, then $\mathcal{E}(0) = |\alpha|^2$, $\mathcal{G} = \alpha^2$, so for $\tau \gg 1$ and $\gamma = 0$ (exact resonance) we have $\mathcal{E}(\tau \gg 1) \approx \frac{1}{4}|\alpha|^2\mathrm{e}^{4\tau}\left[2 - \cos(2\phi)\right]$. The maximum value of the energy in this case is three times larger than the minimum value, depending on the phase ϕ.

According to (3.5), the initial stage of the evolution does not depend on the detuning parameter γ for all states which yield $\mathrm{Im}\,(\mathcal{G}) = 0$, since at $\tau \to 0$ one has

$$\mathcal{E}(\tau) \approx \mathcal{E}(0) - \sigma\,\mathrm{Re}\,(\mathcal{G})p\tau + 2\left[\mathcal{E}(0) + \frac{p^2-1}{24} + \frac{\gamma\sigma}{2}\mathrm{Im}\,(\mathcal{G})\right](p\tau)^2. \tag{3.6}$$

Equation (3.6) is *exact* in the case of $\gamma = 1$. If $\gamma > 1$, then one should replace each function $\sinh(ax)/a$ in (3.5) by its trigonometric counterpart $\sin(\tilde{a}x)/\tilde{a}$, where

$$\tilde{a} = \sqrt{\gamma^2 - 1}. \tag{3.7}$$

In this case the total energy *oscillates* in time with the period $\pi/(p\tilde{a})$, returning to the initial value at the end of each period. For a large detuning $\gamma \gg 1$ the amplitude of oscillations decreases as γ^{-1} if $\mathrm{Re}\mathcal{G} \neq 0$ and as γ^{-2} otherwise. For the initial vacuum state of field we have

$$\mathcal{E}^{(vac)}(\tau) = \frac{p^2-1}{12a^2}\sinh^2(pa\tau)\,. \tag{3.8}$$

The total number of photons in all the modes equals $\mathcal{N} = \mathcal{N}^{(vac)} + \mathcal{N}^{(cav)}$, where

$$\mathcal{N}^{(vac)} = \sum_{m,n=1}^{\infty}\frac{m}{n}|\eta_m^{(n)}|^2 \tag{3.9}$$

is the total number of photons generated from vacuum, and the sum

$$\mathcal{N}^{(cav)} = \mathcal{N}(0) + 2 \sum_{m,n,k=1}^{\infty} \frac{m}{\sqrt{nk}} \left[\eta_m^{(n)*} \eta_m^{(k)} \langle \hat{b}_n^\dagger \hat{b}_k \rangle + \mathrm{Re}\left(\eta_m^{(n)} \xi_m^{(k)} \langle \hat{b}_n \hat{b}_k \rangle \right) \right] \tag{3.10}$$

describes the influence of the initial state of the field (to obtain equation (3.10) one should take into account the identity (2.12)). Differentiating equations (3.9) and (3.10) with respect to τ and performing the summation over m with the help of equations (2.17)–(2.22) or (2.24) one can obtain the formulae

$$\frac{\mathrm{d}\mathcal{N}^{(vac)}}{\mathrm{d}\tau} = 2\sigma \mathrm{Re} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{m=1}^{p} m(p-m) \rho_{-m}^{(n)*}(\tau) \rho_{p-m}^{(n)}(\tau) \tag{3.11}$$

$$\begin{aligned}\frac{\mathrm{d}\mathcal{N}^{(cav)}}{\mathrm{d}\tau} = {} & 2\sigma \sum_{n,k=1}^{\infty} \frac{\langle \hat{b}_n^\dagger \hat{b}_k \rangle}{\sqrt{nk}} \sum_{m=1}^{p} m(p-m) \left[\rho_{-m}^{(n)*} \rho_{p-m}^{(k)} + \rho_{-m}^{(k)} \rho_{p-m}^{(n)*} \right] \\ & -2\sigma \mathrm{Re} \sum_{n,k=1}^{\infty} \frac{\langle \hat{b}_n \hat{b}_k \rangle}{\sqrt{nk}} \sum_{m=1}^{p} m(p-m) \left[\rho_{-m}^{(n)} \rho_{m-p}^{(k)} + \rho_{m}^{(n)} \rho_{p-m}^{(k)} \right].\end{aligned} \tag{3.12}$$

Consequently, to calculate the total number of photons one has to know the coefficients $\eta_m^{(n)}$ and $\xi_m^{(n)}$ with the lower indices $m = 1, 2, \ldots, p-1$.

4. The ‘semi-resonance’ case (p = 1)

Let us start calculating the Bogoliubov coefficients with the ‘semi-resonance’ case $p = 1$. It is distinct, since all the coefficients $\eta_k^{(n)}(t)$ are equal to zero, and the total number of photons is conserved. In this specific case one has to solve the set of equations ($k, n = 1, 2, \ldots$)

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \xi_k^{(n)} = (k-1)\xi_{k-1}^{(n)} - (k+1)\xi_{k+1}^{(n)} + 2\mathrm{i}\gamma k \xi_k^{(n)} \tag{4.1}$$

with the initial conditions $\xi_k^{(n)}(0) = \delta_{kn}$. To get rid of the infinite number of equations we introduce the *generating function*

$$X^{(n)}(z,\tau) = \sum_{k=1}^{\infty} \xi_k^{(n)}(\tau) z^k \tag{4.2}$$

where z is an auxiliary variable. Using the relation $kz^k = z(\mathrm{d}z^k/\mathrm{d}z)$ one obtains the first-order partial differential equation

$$\frac{\partial X^{(n)}}{\partial \tau} = \left(z^2 - 1 + 2\mathrm{i}\gamma z\right) \frac{\partial X^{(n)}}{\partial z} + \xi_1^{(n)}(\tau) \tag{4.3}$$

whose solution satisfying the initial condition $X^{(n)}(0,z) = z^n$ reads

$$X^{(n)}(z,\tau) = \left[\frac{zg(\tau) - S(\tau)}{g^*(\tau) - zS(\tau)} \right]^n + \int_0^\tau \xi_1^{(n)}(x)\, \mathrm{d}x \tag{4.4}$$

where

$$S(\tau) = \sinh(a\tau)/a \qquad g(\tau) = \cosh(a\tau) + \mathrm{i}\gamma S(\tau). \tag{4.5}$$

Differentiating equation (4.4) over z we find

$$\xi_1^{(n)}(\tau) = \frac{n[-S(\tau)]^{n-1}}{[g^*(\tau)]^{n+1}}. \tag{4.6}$$

Putting this expression into the integral on the right-hand side of equation (4.4) we arrive at the final form of the generating function

$$X^{(n)}(z,\tau)=\left[\frac{zg(\tau)-S(\tau)}{g^*(\tau)-zS(\tau)}\right]^n-\left[\frac{-S(\tau)}{g^*(\tau)}\right]^n \tag{4.7}$$

which satisfies automatically the necessary boundary condition $X^{(n)}(\tau,0)=0$. The right-hand side of (4.7) can be expanded into the power series of z with the aid of the formula [44] (volume 3, section 19.6, equation (16))

$$(1-t)^{b-c}(1-t+xt)^{-b}=\sum_{m=0}^{\infty}\frac{t^m}{m!}(c)_m F(-m,b;c;x)$$

where $F(a,b;c;x)$ means the Gauss hypergeometric function, and $(c)_k\equiv\Gamma(c+k)/\Gamma(c)$. In turn, the function $(c)_m F(-m,b;c;x)$ with an integer m is reduced to the Jacobi polynomial in accordance with the formula [44] (volume 2, section 10.8, equation (16))

$$(c)_m F(-m,b;c;x)=m!(-1)^m P_m^{(b-m-c,\,c-1)}(2x-1).$$

Consequently

$$(1-t)^{b-c}(1-t+xt)^{-b}=\sum_{m=0}^{\infty}(-t)^m P_m^{(b-m-c,\,c-1)}(2x-1) \tag{4.8}$$

and the coefficient $\xi_m^{(n)}(\tau)$ reads

$$\xi_m^{(n)}(\tau)=(-\kappa)^{n-m}\lambda^{n+m}P_m^{(n-m,\,-1)}\left(1-2\kappa^2\right) \tag{4.9}$$

where

$$\kappa(\tau)=\frac{S}{\sqrt{gg^*}}\equiv\frac{S(\tau)}{\sqrt{1+S^2(\tau)}} \tag{4.10}$$

$$\lambda(\tau)=\sqrt{g(\tau)/g^*(\tau)}\equiv\sqrt{1-\gamma^2\kappa^2}+\mathrm{i}\gamma\kappa \qquad |\lambda|=1. \tag{4.11}$$

The form (4.9) is useful for $n\geqslant m$. To find a convenient formula in the case of $n\leqslant m$ we introduce the *two-dimensional* generating function

$$X(\tau,z,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}z^m y^n\xi_m^{(n)}(\tau)=\sum_{n=1}^{\infty}X^{(n)}(z,\tau)y^n$$
$$=\frac{yz}{[g^*(\tau)+yS(\tau)][g^*(\tau)-g(\tau)yz+S(\tau)(y-z)]}. \tag{4.12}$$

The coefficient at z^m in (4.12) yields another one-dimensional generating function:

$$X_m(\tau,y)=\sum_{n=1}^{\infty}y^n\xi_m^{(n)}(\tau)=y\frac{[g(\tau)y+S(\tau)]^{m-1}}{[g^*(\tau)+yS(\tau)]^{m+1}}. \tag{4.13}$$

Then equation (4.8) results in the expression

$$\xi_m^{(n)}=(1-\kappa^2)\kappa^{m-n}\lambda^{n+m}P_{n-1}^{(m-n,\,1)}\left(1-2\kappa^2\right). \tag{4.14}$$

Note that the functions $S(\tau)$, $\cosh(a\tau)$ and $\kappa(\tau)$ are real for any value of γ. For $\gamma>1$ it is convenient to use instead of (4.5) the equivalent expressions in terms of the trigonometric functions

$$\tilde{S}(\tau)=\sin(\tilde{a}\tau)/\tilde{a} \qquad \tilde{g}(\tau)=\cos(\tilde{a}\tau)+\mathrm{i}\gamma\tilde{S}(\tau). \tag{4.15}$$

In the special case $\gamma = 1$ one has $S(\tau) = \tau$ and $g(\tau) = 1 + \mathrm{i}\tau$. In particular

$$\xi_m^{(n)}(\tau; \gamma = 1) = \frac{\tau^{m-n}(1+\mathrm{i}\tau)^{n-1}}{(1-\mathrm{i}\tau)^{m+1}} P_{n-1}^{(m-n,\,1)}\left(\frac{1-\tau^2}{1+\tau^2}\right). \tag{4.16}$$

Knowledge of the two-dimensional generating function enables to verify the unitarity condition (2.13). Consider the product $X^*(\tau, z_1, y_1)X(\tau, z_2, y_2)$, which is a four–variable generating function for the products $\xi_m^{(n)*}\xi_l^{(k)}$. Taking $y_1 = \sqrt{u}\exp(\mathrm{i}\varphi)$, $y_2^* = \sqrt{u}\exp(-\mathrm{i}\varphi)$ and integrating over φ from 0 to 2π one obtains a three–variable generating function $\sum z_1^{*m} z_2^l u^n \xi_m^{(n)*}\xi_l^{(n)}$. Dividing it by u and integrating the ratio over u from 0 to 1 one finally arrives at the relation

$$\sum_{n,m,l=1}^{\infty} z_1^{*m} z_2^l \frac{1}{n}\xi_m^{(n)*}\xi_l^{(n)} = -\ln(1 - z_1^* z_2) = \sum_{k=1}^{\infty}\frac{1}{k}(z_1^* z_2)^k \tag{4.17}$$

which is equivalent to the special case of (2.13) for $\eta_m^{(k)} \equiv 0$:

$$\sum_n \frac{1}{n}\xi_m^{(n)*}(\tau)\xi_j^{(n)}(\tau) \equiv \frac{1}{m}\delta_{mj}. \tag{4.18}$$

Suppose that initially there was a single excited mode labeled with an index n. Due to the linearity of the process one may assume that the mean number of photons in this mode was $\nu_n = 1$. Then the mean occupation number of the mth mode at $\tau > 0$ equals

$$\mathcal{N}_m^{(n)} = \frac{m}{n}\left[\xi_m^{(n)}\right]^2 = \frac{m}{n}\left[(1-\kappa^2)\kappa^{m-n} P_{n-1}^{(m-n,\,1)}\left(1 - 2\kappa^2\right)\right]^2 \tag{4.19}$$

where κ is given by (4.10). Although equation (4.19) seems asymmetric with respect to the indices m and n, in fact the relation

$$\mathcal{N}_m^{(n)} = \mathcal{N}_n^{(m)} \tag{4.20}$$

holds. To prove it we calculate the generating function

$$Q(u, v) \equiv \sum_{m,n=1}^{\infty} v^m u^n \mathcal{N}_m^{(n)}. \tag{4.21}$$

It is related to the function $X(z, y)$ (4.12) as follows:

$$Q(u, v) = v\frac{\mathrm{d}}{\mathrm{d}v}\int_0^u \mathrm{d}r \int_0^{2\pi}\int_0^{2\pi}\frac{\mathrm{d}\varphi\,\mathrm{d}\psi}{(2\pi)^2} X\left(\sqrt{r}\mathrm{e}^{\mathrm{i}\varphi}, \sqrt{v}\mathrm{e}^{\mathrm{i}\psi}\right) X^*\left(\sqrt{r}\mathrm{e}^{\mathrm{i}\varphi}, \sqrt{v}\mathrm{e}^{\mathrm{i}\psi}\right).$$

Having performed all the calculations we arrive at the expression

$$2Q(u, v) = \frac{1 + uv - \kappa^2(u+v)}{\left\{\left[1 + uv - \kappa^2(u+v)\right]^2 - 4uv(1-\kappa^2)^2\right\}^{1/2}} - 1. \tag{4.22}$$

Then equation (4.20) is a consequence of the relation $Q(u, v) = Q(v, u)$.

The initial stage of the evolution of $\mathcal{N}_m^{(n)}(\tau)$ does not depend on the detuning parameter γ, since the principal term of the expansion of (4.19) with respect to τ yields

$$\mathcal{N}_{n\pm q}^{(n)}(\tau \to 0) = \frac{n \pm q}{n}\left[\frac{n(n\pm 1)\cdots(n \pm q \mp 1)}{q!}\right]^2 \tau^{2q}.$$

However, the further evolution is sensitive to the value of γ. If $\gamma \leqslant 1$, then the function $\mathcal{N}_m^{(n)}(\tau)$ has many maxima and minima (especially for large values of m and n), but finally it decreases asymptotically as $mna^4/\cosh^4(a\tau)$. In contrast, if $\gamma > 1$, then the function $\mathcal{N}_m^{(n)}(\tau)$ is periodic with the period $\pi/\tilde{a}$, and it turns into zero for $\tau = k\pi/\tilde{a}$,

$k = 1, 2, \ldots$ (excepting the case $m = n$). The magnitude of the coefficient $\mathcal{N}_m^{(n)}(\tau)$ decreases approximately as $\gamma^{-2|m-n|}$ for $\gamma \gg 1$.

In the special case of a cavity filled in with a *high-temperature thermal radiation*, the initial distribution over modes reads $\nu_n(T) = T/n$, constant T being proportional to the temperature. Then $\mathcal{N}_m^{\{T\}} = \sum_n \nu_n(T)\mathcal{N}_m^{(n)}$. This sum is simply T multiplied by the coefficient at v^m in the Taylor expansion of the function

$$\tilde{Q}(v) = \int_0^1 \frac{\mathrm{d}u}{u} Q(u, v) = \ln \frac{1 - v\kappa^2(\tau)}{1 - v}.$$

Thus we have

$$\mathcal{E}_m^{\{T\}} = m\mathcal{N}_m^{\{T\}} = T\big(1 - [\kappa(\tau)]^{2m}\big).$$

We see that the resonance vibrations of the wall cause an effective cooling of the lowest electromagnetic modes (provided $|\gamma| < 1$). The total number of quanta and the total energy in this example are formally infinite, due to the equipartition law of the classical statistical mechanics. In reality both these quantities are finite, since $\nu_n(T) < T/n$ at $n \to \infty$ due to the quantum corrections. Other initial conditions in the special case of the *exact* resonance ($\gamma = 0$) were considered in [22]. The total energy depends on time according to equation (3.5) with $p = 1$. An infinite growth of the energy of a classical string whose ends oscillate at the frequency close to ω_1 in the case of finite amplitude and detuning ($\varepsilon \sim \delta \sim \mathcal{O}(1)$) was considered in [45].

5. The generic resonance case $p \geqslant 2$

Now we turn to calculating the non-zero Bogoliubov coefficients $y_m^{(n,j)}(\tau)$, equations (2.30), in the generic case $p \geqslant 2$. One can easily verify that in the distinct case $j = 0$ the functions $y_m^{(n,0)}(\tau)$ with $m \geqslant 1$ are given by the formulae for $\xi_m^{(n)}(\tau)$ found in section 4, provided one replaces τ by $\sigma p\tau$ and γ by $\sigma\gamma$ (recall that $\sigma \equiv (-1)^p$), whereas $y_m^{(n,0)}(\tau) \equiv 0$ for $m \leqslant 0$. In the generic case $j \neq 0$ it is reasonable to introduce a generating function in the form of the *Laurent series* of an auxiliary variable z

$$R^{(n,j)}(z, \tau) = \sum_{m=-\infty}^{\infty} y_m^{(n,j)}(\tau) z^m \tag{5.1}$$

since the lower index of the coefficient $y_m^{(n,j)}$ runs over all integers from $-\infty$ to ∞. One can verify that the function (5.1) satisfies the *homogeneous* equation

$$\frac{\partial R^{(n,j)}}{\partial \tau} = \left[\sigma\left(\frac{1}{z} - z\right) + 2\mathrm{i}\gamma\right]\left(j + pz\frac{\partial}{\partial z}\right) R^{(n,j)}. \tag{5.2}$$

The solution to (5.2) satisfying the initial condition $R^{(n,j)}(z, 0) = z^n$ reads

$$R^{(n,j)}(z, \tau) = z^{-j/p}\left[\frac{zg(p\tau) + \sigma S(p\tau)}{g^*(p\tau) + z\sigma S(p\tau)}\right]^{n+j/p} \tag{5.3}$$

where the functions $S(\tau)$ and $g(\tau)$ were defined in (4.5). The coefficients of the Laurent series (5.1) can be calculated with the aid of the Cauchy formula

$$y_m^{(n,j)}(\tau) = \frac{1}{2\pi\mathrm{i}} \oint_{\mathcal{C}} \frac{\mathrm{d}z}{z^{m+1}} R^{(n,j)}(z, \tau) \tag{5.4}$$

where the closed curve $\mathcal{C}$ rounds the point $z = 0$ in the complex plane in the anticlockwise direction. Making a scale transformation one can reduce the integral (5.4) with the integrand

(5.3) to the integral representation of the Gauss hypergeometric function [44] (volume 1, section 2.1.3)

$$F(a,b;c;x) = \frac{-\mathrm{i}\Gamma(c)\exp(-\mathrm{i}\pi b)}{2\sin(\pi b)\Gamma(c-b)\Gamma(b)} \oint_1^{(0+)} \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tx)^a}\,\mathrm{d}t \tag{5.5}$$

where $\mathrm{Re}\,(c-b) > 0$, $b \neq 1, 2, 3, \ldots$, and the integration contour begins at the point $t = 1$ and passes around the point $t = 0$ in the positive direction. After some algebra one can obtain the expression

$$y_m^{(n,j)} = -\frac{\Gamma(-m-j/p)\Gamma(1+n+j/p)\sin\big[\pi(m+j/p)\big]}{\pi\Gamma(1+n-m)} \times (\sigma\kappa)^{n-m}\lambda^{m+n+2j/p} F\big(n+j/p\,,\ -m-j/p\,;\ 1+n-m\,;\ \kappa^2\big). \tag{5.6}$$

Hereafter we assume $\kappa \equiv \kappa(p\tau)$ and $\lambda \equiv \lambda(p\tau)$, the functions $\kappa(x)$ and $\lambda(x)$ being defined as in (4.10) and (4.11). Using the known formula

$$\Gamma(-z)\sin(\pi z) = -\pi/\Gamma(z+1) \tag{5.7}$$

one can eliminate the gamma function of a negative argument:

$$y_m^{(n,j)} = \frac{\Gamma(1+n+j/p)(\sigma\kappa)^{n-m}\lambda^{m+n+2j/p}}{\Gamma(1+m+j/p)\Gamma(1+n-m)} F\big(n+j/p\,,\ -m-j/p\,;\ 1+n-m\,;\ \kappa^2\big). \tag{5.8}$$

The form (5.8) gives an explicit expression for the coefficient $\xi_{j+pm}^{(j+pn)}$ with $0 \leqslant m \leqslant n$. Moreover, it clearly shows the fulfilment of the initial condition $y_m^{(n,j)}(\tau = 0) = \delta_{mn}$. Transforming the hypergeometric function with the aid [44, 46] of the formula

$$\lim_{c\to -n} \frac{F(a,b;c;x)}{\Gamma(c)} = \frac{(a)_{n+1}(b)_{n+1}x^{n+1}}{(n+1)!} F(a+n+1, b+n+1; n+2; x)$$

$(n = 0, 1, 2, \ldots)$ and the identity (5.7) one obtains an equivalent expression

$$y_m^{(n,j)} = \frac{\Gamma(m+j/p)(-\sigma\kappa)^{m-n}\lambda^{m+n+2j/p}}{\Gamma(n+j/p)\Gamma(1+m-n)} F\big(m+j/p\,,\ -n-j/p\,;\ 1+m-n\,;\ \kappa^2\big) \tag{5.9}$$

which gives a convenient form of the coefficient $\xi_{j+pm}^{(j+pn)}$ for $m \geqslant n$. Equation (5.6) with negative values of the lower index gives an explicit expression for the non-zero coefficients $\eta_{pk-j}^{(pn+j)}$ $(k \geqslant 1, n \geqslant 0)$:

$$\eta_{pk-j}^{(pn+j)} = -\frac{\Gamma(k-j/p)\Gamma(1+n+j/p)\sin\big[\pi(k-j/p)\big]}{\pi\Gamma(1+n+k)} \times (\sigma\kappa)^{n+k}\lambda^{n-k+2j/p} F\big(n+j/p\,,\ k-j/p\,;\ 1+n+k\,;\ \kappa^2\big). \tag{5.10}$$

Note that expressions (5.8)–(5.10) are also valid for $j = 0$. In this case they coincide with the formulae obtained in section 4. Equations (5.8)–(5.10) immediately give the short-time behaviour of the Bogoliubov coefficients at $\tau \to 0$: it is sufficient to put $\kappa \approx p\tau$, $\lambda \approx 1$ and to replace the hypergeometric functions by 1. In this limit the detuning parameter γ drops out of the expressions (in the leading terms of the Taylor expansions).

At $\tau \to \infty$ we have the following asymptotics of the functions $\kappa(p\tau)$ and $\lambda(p\tau)$ (if $\gamma \leqslant 1$)

$$\kappa \approx 1 - \tfrac{1}{2}S^{-2}(p\tau) \to 1 \qquad \lambda \to a + \mathrm{i}\gamma \qquad \tau \to \infty.$$

Then equation (5.6), together with the known asymptotics of the hypergeometric function $F(a,b;a+b+1;1-x)$ at $x \ll 1$ [44, 46]

$$F(a,b;a+b+1;1-x) = \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b+1)}\left[1+abx\ln(x)+\mathcal{O}(x)\right] \tag{5.11}$$

lead to the asymptotic expression for the Bogoliubov coefficients

$$y_m^{(n,j)}(\tau \gg 1) = \frac{\sin[\pi(m+j/p)]}{\pi(m+j/p)}(a+\mathrm{i}\gamma)^{m+n+2j/p}\sigma^{n-m}\left[1+\mathcal{O}\left(\frac{mn}{S^2}\ln S\right)\right]. \tag{5.12}$$

For $\gamma < 1$ the correction is of order $mn\tau\exp(-2ap\tau)$, while for $\gamma = 1$ it is of order $mn\ln(\tau)/\tau^2$.

One can verify that the generating function (5.3) satisfies the recurrence relation

$$\frac{\partial R^{(q,j)}}{\partial\tau} = (j+qp)\left\{\sigma\left[R^{(q-1,j)} - R^{(q+1,j)}\right] + 2\mathrm{i}\gamma R^{(q,j)}\right\}. \tag{5.13}$$

Its immediate consequence is an analogous relation for the Bogoliubov coefficients with the same lower indices:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\rho_m^{(n)} = n\left\{\sigma\left[\rho_m^{(n-p)} - \rho_m^{(n+p)}\right] + 2\mathrm{i}\gamma\rho_m^{(n)}\right\}. \tag{5.14}$$

Equation (5.14) is valid for $n > p$ (when $q \geqslant 1$ and $j \geqslant 1$ in (5.13)), since the coefficients $\rho_m^{(n)}$ are not defined when $n < 0$. However, using the chain of identities

$$\begin{aligned} R^{(-1,j)}(z) &= z^{-j/p}\left[\frac{S+gz}{g^*+Sz}\right]^{j/p-1} = \frac{1}{z}\left(\frac{1}{z}\right)^{j/p-1}\left[\frac{S+g^*/z}{g+S/z}\right]^{1-j/p} \\ &= \frac{1}{z}\left[R^{(0,p-j)}(1/z^*)\right]^* = \frac{1}{z}\sum_{k=-\infty}^{\infty} y_k^{(0,p-j)*}\left(\frac{1}{z}\right)^k = \sum_{k=-\infty}^{\infty} y_{-k-1}^{(0,p-j)*} z^k \end{aligned}$$

one can obtain the first $p-1$ recurrence relations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\rho_m^{(n)} = n\left\{\sigma\left[\rho_{-m}^{(p-n)*} - \rho_m^{(p+n)}\right] + 2\mathrm{i}\gamma\rho_m^{(n)}\right\} \qquad n = 1,2,\ldots,p-1. \tag{5.15}$$

To treat the special case $n = p$ (it corresponds to the distinguished subset with $j = 0$) one should take into account that $R^{(0,0)}(z) \equiv 1$, which means formally that $\rho_m^{(0)} = \delta_{m0}$. So the last recurrence relation reads

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\rho_m^{(p)} = p\left\{-\sigma\rho_m^{(2p)} + 2\mathrm{i}\gamma\rho_m^{(p)}\right\} \qquad m \geqslant 1$$

(recall that $\rho_m^{(p)} \equiv 0$ for $m \leqslant 0$). Now one can verify that the unitarity conditions (2.27), (2.28) are the consequencies of the equations (5.14) and (5.15).

Differentiating the 'vacuum' part of sum (2.15) with respect to τ and performing the summation over the upper index n with the aid of (5.14), (5.15) (recalling that the coefficients $\rho_m^{(n)}$ are different from zero provided the difference $n-m$ is a multiple of p), one can obtain the formula for the photon generation rate from vacuum in each mode ($0 \leqslant j \leqslant p-1$, $q = 0,1,2,\ldots$)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{N}_{j+pq}^{(vac)} &= -2\sigma(j+pq)\mathrm{Re}\left[\xi_{j+pq}^{(j)}\eta_{j+pq}^{(p-j)}\right] \\ &= 2p\sqrt{1-\gamma^2\kappa^2}\,\frac{\sin(\pi j/p)\Gamma(q+j/p)\Gamma(1+q+j/p)\Gamma(2-j/p)}{\pi\Gamma(j/p)\Gamma(q+1)\Gamma(q+2)}\kappa^{2q+1} \\ &\quad\times F\left(q+j/p\,,\,-j/p\,;\,1+q\,;\,\kappa^2\right)F\left(q+j/p\,,\,1-j/p\,;\,2+q\,;\,\kappa^2\right). \end{aligned} \tag{5.16}$$

We see that there is no photon creation in the modes with numbers $p, 2p, \ldots$. At $\tau \ll 1$ we have $\dot{\mathcal{N}}^{(vac)}_{j+pq} \sim \tau^{2q+1}$. In the long-time limit the photon generation rate tends to the constant value (if $\gamma < 1$)

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{N}^{(vac)}_{j+pq} = \frac{2ap^2 \sin^2(\pi j/p)}{\pi^2(j+pq)}\left[1 + \mathcal{O}\left(\frac{pq}{S^2}\ln S\right)\right] \qquad ap\tau \gg 1. \tag{5.17}$$

For $q \gg 1$ and for a fixed value of κ one can simplify the right-hand side of (5.16) using Stirling's formula for the Gamma functions and the easily verified asymptotic formula

$$F(a, b; c; z) \approx (1 - az/c)^{-b} \qquad a, c \gg 1.$$

In this case

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{N}^{(vac)}_{j+pq} \approx 2p\sqrt{1-\gamma^2\kappa^2}\,\frac{\sin(\pi j/p)\Gamma(2-j/p)\kappa^{2q+1}}{\pi\Gamma(j/p)q^{2(1-j/p)}\left(1-\kappa^2\right)^{1-2j/p}} \qquad q \gg 1. \tag{5.18}$$

In particular, if $q \gg S^2(p\tau) \gg 1$, then

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\mathcal{N}^{(vac)}_{j+pq} \approx 2pa\,\frac{\sin(\pi j/p)\Gamma(2-j/p)(S^2/q)^{2(1-j/p)}}{\pi\Gamma(j/p)S^2}\exp(-q/S^2). \tag{5.19}$$

Comparing (5.17) and (5.19), one can conclude that the number of the effectively excited modes (i.e. the modes with a time-independent photon generation rate) increases in time exponentially, approximately as $S^2(\tau)/\ln S(\tau)$.

Differentiating equation (3.11) once again over τ, one can perform the summation over the upper index n with the aid of equations (5.14), (5.15) to obtain a closed expression for the *second derivative* of the total number of 'vacuum' photons:

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\mathcal{N}^{(vac)} = 2\mathrm{Re}\sum_{m=1}^{p-1} m(p-m)\left[\xi_m^{(m)}\xi_{p-m}^{(p-m)} + \eta_m^{(p-m)*}\eta_{p-m}^{(m)*}\right]$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\mathcal{N}^{(vac)} = 2\sum_{m=1}^{p-1} m(p-m)\left\{m(p-m)\left[\frac{\kappa}{p}F\left(\frac{m}{p}\,,\ 1-\frac{m}{p}\,;\ 2\,;\ \kappa^2\right)\right]^2\right.$$
$$\left. + \left(1-2\gamma^2\kappa^2\right)F\left(\frac{m}{p}\,,\ -\frac{m}{p}\,;\ 1\,;\ \kappa^2\right)F\left(\frac{m}{p}-1\,,\ 1-\frac{m}{p}\,;\ 1\,;\ \kappa^2\right)\right\}. \tag{5.20}$$

In the short-time limit one obtains

$$\ddot{\mathcal{N}}^{(vac)} = \tfrac{1}{3}p(p^2-1) \qquad |ap\tau| \ll 1. \tag{5.21}$$

In the long-time limit the equations (5.7), (5.11) and $\sum_{m=1}^{p-1}\sin^2(\pi m/p) = p/2$ lead to another simple expression (provided $p \geqslant 2$)

$$\ddot{\mathcal{N}}^{(vac)} = 2a^2p^3/\pi^2 \qquad ap\tau \gg 1, \quad a > 0. \tag{5.22}$$

Consequently, the total number of photons created from vacuum due to NSCE increases in time quadratically both in the short-time and in the long-time limits (although with different coefficients).

It is interesting to compare equation (5.22) with the total rate of change of the number of 'cavity' photons due to non-vacuum initial conditions. Using equation (3.12) and replacing the coefficients $\rho_m^{(n)}$ by their asymptotic values (5.12) one can obtain the expression

$$\frac{\mathrm{d}\mathcal{N}^{(cav)}}{\mathrm{d}\tau} = \frac{4ap^2}{\pi^2}\sum_{m=1}^{p-1}\sin^2(\pi m/p)\sum_{n,k=0}^{\infty}\frac{\sigma^{n+k}}{\sqrt{(m+pn)(m+pk)}}$$
$$\times\left\{\langle\hat{b}^{\dagger}_{m+pn}\hat{b}_{m+pk}\rangle(a+\mathrm{i}\gamma)^{k-n} - \sigma\mathrm{Re}\left[\langle\hat{b}_{m+pn}\hat{b}_{m+pk}\rangle(a+\mathrm{i}\gamma)^{k+n+1}\right]\right\} \tag{5.23}$$

which holds provided $ap\tau \gg 1$ and $a > 0$. For the physical initial states the sum on the right-hand side of (5.23) is finite. This is obvious if a finite number of modes was excited initially. But even if the cavity was initially in a high-temperature thermal state, so that $\langle \hat{b}_n^\dagger \hat{b}_k \rangle = \delta_{nk} T/n$, $\langle \hat{b}_n \hat{b}_k \rangle = 0$, the sum over n, k yields a finite value $T \sum_{n=0}^{\infty} (m + pn)^{-2}$. Consequently, the total number of 'non-vacuum' photons increases in time *linearly* at $ap\tau \gg 1$, whereas the total number of quanta generated from vacuum increases *quadratically* in the long time limit. At the same time, the total 'vacuum' and 'non-vacuum' energies increase exponentially if $\gamma < 1$ (see section 3). The origin of the difference in the behaviours of the total energy and the total number of photons becomes clear, if one looks at the asymptotic formulae (5.17)–(5.19). They show that the rate of photon generation in the mth completely excited mode decreases approximately as $1/m$ (excepting the modes whose numbers are multiples of p), so the stationary rate of the *energy* generation asymptotically almost does not depend on m. In turn, the number of the effectively excited modes increases in time exponentially. These two factors lead to the exponential growth of the total energy (see also [27] in the special case $\gamma = 0$).

6. The 'principal resonance' case ($p = 2$)

Some of the formulae obtained in the preceding section can be simplified in the special case $p = 2$. In this case there are two subsets of non-zero Bogoliubov coefficients. The first one consists of the coefficients with even upper and lower indices $\xi_{2k}^{(2q)}$ which are reduced to the coefficients $\xi_k^{(q)}$ of the 'semi-resonance' case (since $\eta_{2k}^{(2q)} \equiv 0$, this subset does not contribute to the generation of new photons). The second subset is formed by the 'odd' coefficients which can be written ($\kappa \equiv \kappa(2\tau)$) as

$$\xi_{2m+1}^{(2n+1)} = \frac{\Gamma(n+3/2)\kappa^{n-m}\lambda^{m+n+1}}{\Gamma(m+3/2)\Gamma(1+n-m)} \times F\big(n+1/2\,,\ -m-1/2\,;\ 1+n-m\,;\ \kappa^2\big) \qquad n \geqslant m \tag{6.1}$$

$$\xi_{2m+1}^{(2n+1)} = \frac{(-1)^{m-n}\Gamma(m+1/2)\kappa^{m-n}\lambda^{m+n+1}}{\Gamma(n+1/2)\Gamma(1+m-n)} \times F\big(m+1/2\,,\ -n-1/2\,;\ 1+m-n\,;\ \kappa^2\big) \qquad m \geqslant n \tag{6.2}$$

$$\eta_{2k+1}^{(2n+1)} = \frac{(-1)^{k-1}\Gamma(k+1/2)\Gamma(n+3/2)\kappa^{n+k+1}\lambda^{n-k}}{\pi\Gamma(2+n+k)} \times F\big(n+1/2\,,\ k+1/2\,;\ 2+n+k\,;\ \kappa^2\big). \tag{6.3}$$

All the 'odd' coefficients can be expressed [47] in terms of the complete elliptic integrals

$$\mathbf{K}(\kappa) = \int_0^{\pi/2} \frac{\mathrm{d}\alpha}{\sqrt{1-\kappa^2\sin^2\alpha}} \qquad \mathbf{E}(\kappa) = \int_0^{\pi/2} \mathrm{d}\alpha\sqrt{1-\kappa^2\sin^2\alpha}.$$

In particular

$$\xi_1^{(1)} = \frac{2}{\pi}\lambda(\kappa)\mathbf{E}(\kappa) \qquad \eta_1^{(1)} = \frac{2}{\pi\kappa}\big[\tilde{\kappa}^2\mathbf{K}(\kappa) - \mathbf{E}(\kappa)\big] \tag{6.4}$$

where

$$\tilde{\kappa} \equiv \sqrt{1-\kappa^2} = \big[1 + S^2(2\tau)\big]^{-1/2}. \tag{6.5}$$

However, the analogous expressions for the coefficients $\xi_m^{(n)}$ and $\eta_m^{(n)}$ with $m, n > 1$ appear rather cumbersome (they can be written as linear combinations of the functions $\mathbf{E}(\kappa)$ and $\mathbf{K}(\kappa)$ multiplied by some rational functions of κ and $\tilde{\kappa}$), so we do not bring them here.

The photon generation rate from vacuum in the principal cavity mode ($m = 1$) reads

$$\frac{\mathrm{d}\mathcal{N}_1^{(vac)}}{\mathrm{d}\tau} = -2\mathrm{Re}\left[\eta_1^{(1)}\xi_1^{(1)}\right] = \frac{8\sqrt{1-\gamma^2\kappa^2}}{\pi^2\kappa}\mathbf{E}(\kappa)\left[\mathbf{E}(\kappa) - \tilde{\kappa}^2\mathbf{K}(\kappa)\right]. \qquad (6.6)$$

The total number of photons in the first mode can be obtained by integrating equation (6.6). Taking into account the relation

$$\sqrt{1-\gamma^2\kappa^2}\,\mathrm{d}\tau = \mathrm{d}\kappa/\tilde{\kappa}^2 \qquad (6.7)$$

and the differentiation rules for the complete elliptic integrals

$$\frac{\mathrm{d}\mathbf{K}(\kappa)}{\mathrm{d}\kappa} = \frac{\mathbf{E}(\kappa)}{\kappa\tilde{\kappa}^2} - \frac{\mathbf{K}(\kappa)}{\kappa} \qquad \frac{\mathrm{d}\mathbf{E}(\kappa)}{\mathrm{d}\kappa} = \frac{\mathbf{E}(\kappa) - \mathbf{K}(\kappa)}{\kappa} \qquad (6.8)$$

one can verify the following result:

$$\mathcal{N}_1^{(vac)}(\kappa) = \frac{2}{\pi^2}\mathbf{K}(\kappa)\left[2\mathbf{E}(\kappa) - \tilde{\kappa}^2\mathbf{K}(\kappa)\right] - \frac{1}{2}. \qquad (6.9)$$

Making the transformation [44, 46]

$$\mathbf{K}\left(\frac{1-\tilde{\kappa}}{1+\tilde{\kappa}}\right) = \frac{1+\tilde{\kappa}}{2}\mathbf{K}(\kappa) \qquad \mathbf{E}\left(\frac{1-\tilde{\kappa}}{1+\tilde{\kappa}}\right) = \frac{\mathbf{E}(\kappa) + \tilde{\kappa}\mathbf{K}(\kappa)}{1+\tilde{\kappa}}$$

one can rewrite equations (6.4) and (6.9) in the form given in [21] for $\gamma = 0$. Using the asymptotic expansions of the elliptic integrals [48] at $\kappa \to 1$

$$\mathbf{K}(\kappa) \approx \ln\frac{4}{\tilde{\kappa}} + \frac{1}{4}\left(\ln\frac{4}{\tilde{\kappa}} - 1\right)\tilde{\kappa}^2 + \cdots$$

$$\mathbf{E}(\kappa) \approx 1 + \frac{1}{2}\left(\ln\frac{4}{\tilde{\kappa}} - \frac{1}{2}\right)\tilde{\kappa}^2 + \cdots$$

one can obtain the formula

$$\mathcal{N}_1^{(vac)}(\tau \gg 1) = \frac{8a}{\pi^2}\tau + \frac{4}{\pi^2}\ln\left(\frac{2}{a}\right) - \frac{1}{2} + \mathcal{O}\left(\tau\,\mathrm{e}^{-4a\tau}\right) \qquad a > 0. \qquad (6.10)$$

In the special case of $\gamma = 1$ one can obtain the expansion

$$\mathcal{N}_1^{(vac)}(\tau \gg 1) = \frac{4}{\pi^2}\ln\tau + \frac{12}{\pi^2}\ln 2 - \frac{1}{2} + \mathcal{O}(\tau^{-2}).$$

If $\gamma > 1$, the number of photons in the principal mode oscillates with the period $\pi/(2\tilde{a})$. For $\gamma \gg 1$ one can write $\kappa \approx \sin(2\tilde{a}\tau)/\tilde{a}$, i.e. $|\kappa| \ll 1$. In this case

$$\mathcal{N}_1^{(vac)} \approx \frac{\kappa^2}{4} \approx \frac{\sin^2(2\tilde{a}\tau)}{4\tilde{a}^2} \ll 1.$$

The second derivative of the total number of 'vacuum' photons can be written as

$$\begin{aligned}\frac{\mathrm{d}^2\mathcal{N}^{(vac)}}{\mathrm{d}\tau^2} &= 2\left[\mathrm{Re}\left(\left[\xi_1^{(1)}\right]^2\right) + \left|\eta_1^{(1)}\right|^2\right] \\ &= \frac{8}{\pi^2\kappa^2}\left[\tilde{\kappa}^4\mathbf{K}^2(\kappa) - 2\tilde{\kappa}^2\mathbf{K}(\kappa)\mathbf{E}(\kappa) + \left(1+\kappa^2-2\gamma^2\kappa^4\right)\mathbf{E}^2(\kappa)\right]. \qquad (6.11)\end{aligned}$$

In the limiting cases this formula yields

$$\mathcal{N}^{(vac)}(\tau \ll 1) \approx \tau^2$$

$$\mathcal{N}^{(vac)}(\tau \gg 1) = 8a^2\tau^2/\pi^2 + \mathcal{O}(\tau) \qquad a > 0.$$

If $\gamma \gg 1$, then $|\kappa| \ll 1$, but $\gamma^2\kappa^2 \approx \sin^2(2\tilde{a}\tau) \sim \mathcal{O}(1)$. In this case the Taylor expansion of the expression (6.11) yields $\ddot{\mathcal{N}}^{(vac)} = 2\cos(4\tilde{a}\tau) + \mathcal{O}(\gamma^{-2})$. Integrating this equation taking account of the initial conditions $\dot{\mathcal{N}}^{(vac)}(0) = \mathcal{N}^{(vac)}(0) = 0$, one obtains $\mathcal{N}^{(vac)} \approx \mathcal{N}_1^{(vac)} \approx \sin^2(2\tilde{a}\tau)/(4\tilde{a}^2)$.

7. Discussion

Let us briefly discuss the main results of the paper. We have solved the problem of the photon generation due to the non-stationary Casimir effect in an ideal Fabry–Perot cavity with an equidistant spectrum, if the cavity walls perform small (quasi)resonance oscillations at the frequency $\omega_w = p(\pi c/L_0)(1+\delta)$, for any integer value of $p = 1, 2, \ldots$. Namely, we have found explicit analytical expressions for the Bogoliubov coefficients, the rate of photon production in each mode and the total energy in the case of an arbitrary (although small compared with ω_w) detuning. These expressions are *exact* consequences of the reduced equations (2.24) or (5.14), (5.15). One should remember, however, that the reduced equations arise after averaging the exact equation (2.7) over fast oscillations and neglecting second-order terms with respect to the small parameters ε and δ. Consequently, the 'true' functions $\mathcal{N}(t)$, $\mathcal{E}(t)$, etc. could differ from those given above in terms proportional to ε^2. But such a difference seems quite insignificant under realistic conditions. As was shown in [18, 21], it is hardly possible to obtain the value of the dimensionless amplitude of the *resonance* wall vibrations ε exceeding 10^{-8} in a laboratory. This means that the relative difference between the 'true' magnitude of the photon generation rate (for example) and that given in section 5 could be of the order of 10^{-8} (or less) for $t < t_c \sim (\omega_1\varepsilon^2)^{-1}$. For $\omega_1 \sim 10^{10}$ s^{-1} the characteristic time t_c is of the order of months or years, and even for optical frequences it has an order of seconds (although it is unclear as to how to cause the wall to vibrate at an optical frequency with a sufficiently large amplitude). Another argument in favour of the solutions obtained is that these solutions satisfy *exactly* the Bogoliubov transformation unitarity conditions (2.12)–(2.14).

Note that the rate of photon generation from vacuum in some mode is proportional to $p^2\varepsilon$ (if $\gamma = 0$), and the total generation rate is proportional to $p^3\varepsilon^2$. Actually, the dimensionless amplitude of the wall oscillations ε is inversly proportional to the frequency, since it is determined by the maximum possible stresses inside the wall [18, 21]. Thus we see that increasing the resonance frequency one could achieve, in principle, some amplification of the number of photons proportional to p.

It was shown in the previous studies [10, 14–23] that the photon production from vacuum due to the NSCE *could* be observed under the condition of the strict parametric resonance. Here it is demonstrated explicitly that the photons *cannot* be produced if the detuning δ exceeds the dimensionless amplitude ε. This result confirms once again the statement made in [21] that the NSCE could be observed only in the resonance regime, ruling out the non-resonance laws of motion of the wall. The requirements for a possible experiment turn out to be rather demanding (for example, for the principal frequency about 10 GHz the detuning should not exceed 100 Hz for the time of the order of at least 0.01 s), but they do not seem to be absolutely unrealizable.

Another source of trouble is connected with the non-ideality of real cavities. Until now only a few attempts have been made to take into account different losses in the cavities with moving boundaries [23, 38, 39], and this problem is still a challenge for theoreticians.

Appendix

Using equations (2.15) and (2.23) one can express the total energy in all the modes as

$$\mathcal{E} = \sum_{n=1}^{\infty} \frac{1}{n} S^{(n)} + \sum_{n,k=1}^{\infty} \frac{\langle \hat{b}_n^{\dagger} \hat{b}_k \rangle}{\sqrt{nk}} U_1^{(nk)} + \mathrm{Re} \sum_{n,k=1}^{\infty} \frac{\langle \hat{b}_n \hat{b}_k \rangle}{\sqrt{nk}} U_2^{(nk)}$$

where

$$S^{(n)} = \sum_{m=1}^{\infty} m^2 \left| \rho_{-m}^{(n)} \right|^2 \tag{A.1}$$

$$U_1^{(nk)} = \sum_{m=-\infty}^{\infty} m^2 \rho_m^{(n)*} \rho_m^{(k)} \qquad U_2^{(nk)} = - \sum_{m=-\infty}^{\infty} m^2 \rho_m^{(n)} \rho_{-m}^{(k)} \tag{A.2}$$

(to write $U_2^{(nk)}$ as a sum from $-\infty$ to ∞ one should take into account that the summand in the last sum of (2.15) is symmetrical with respect to n and k). Differentiating $U_1^{(nk)}$ with respect to τ and taking into account equations (2.24), after simple algebra one can obtain the expression

$$\frac{\mathrm{d}}{\mathrm{d}\tau} U_1^{(nk)} = -p(-1)^p \sum_{m=-\infty}^{\infty} m(m+p) \left[\rho_m^{(n)*} \rho_{m+p}^{(k)} + \rho_m^{(k)} \rho_{m+p}^{(n)*} \right]. \tag{A.3}$$

Differentiating the above expression once more, one obtains

$$\ddot{U}_1^{(nk)} = 4p^2 U_1^{(nk)} + 2\mathrm{i}\gamma p^2 (-1)^p \chi_1^{(nk)}$$

where

$$\chi_1^{(nk)} = \sum_{m=-\infty}^{\infty} m(m+p) \left[\rho_m^{(k)} \rho_{m+p}^{(n)*} - \rho_m^{(n)*} \rho_{m+p}^{(k)} \right].$$

Differentiating $\chi_1^{(nk)}$ one can verify that

$$\dot{\chi}_1^{(nk)} = 2\mathrm{i}\gamma (-1)^p \dot{U}_1^{(nk)}.$$

Consequently

$$\chi_1^{(nk)} = 2\mathrm{i}\gamma (-1)^p U_1^{(nk)} + \text{constant}$$

where the additive constant is determined by the initial conditions. Finally we arrive at the equation

$$\ddot{U}_1^{(nk)} = 4p^2 (1-\gamma^2) U_1^{(nk)} + 4p^2 \gamma^2 U_1^{(nk)}(0) + 2\mathrm{i}\gamma p^2 (-1)^p \chi_1^{(nk)}(0)$$

with

$$U_1^{(nk)}(0) = n^2 \delta_{nk} \qquad \chi_1^{(nk)}(0) = nk \left[\delta_{k,n-p} - \delta_{n,k-p} \right].$$

Using the same scheme one can obtain analogous relations for the coefficient $U_2^{(nk)}$:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} U_2^{(nk)} = p(-1)^p \sum_{m=-\infty}^{\infty} m \rho_m^{(n)} \left[(m+p) \rho_{-m-p}^{(k)} - (p-m) \rho_{p-m}^{(k)} \right]$$

$$\ddot{U}_2^{(nk)} = 4p^2 U_2^{(nk)} - 2\mathrm{i}\gamma p^2 (-1)^p \chi_2^{(nk)}$$

$$\chi_2^{(nk)} = \sum_{m=-\infty}^{\infty} m \rho_m^{(n)} \left[(m+p) \rho_{-m-p}^{(k)} + (p-m) \rho_{p-m}^{(k)} \right]$$

$$\dot{\chi}_2^{(nk)} = -2\mathrm{i}\gamma(-1)^p \dot{U}_2^{(nk)}$$
$$\ddot{U}_2^{(nk)} = 4p^2(1-\gamma^2)U_2^{(nk)} - 2\mathrm{i}\gamma p^2(-1)^p \chi_2^{(nk)}(0)$$
$$\chi_2^{(nk)}(0) = nk\delta_{k,p-n}. \tag{A.4}$$

The calculation of the vacuum contribution to the total energy

$$\mathcal{E}^{(vac)} = \sum_{n=1}^{\infty} \frac{1}{n} S^{(n)}$$

is more involved, since the summation in (A.1) is non longer performed from $-\infty$ to ∞, but over the coefficients $\rho_m^{(n)}$ with *negative* indices m only. Differentiating the sum (A.1) with respect to τ and using equations (2.24) we obtain

$$\dot{S}^{(n)} = 2(-1)^p \mathrm{Re} \sum_{m=1}^{\infty} m^2 \rho_{-m}^{(n)} \left[(m+p)\rho_{-m-p}^{(n)*} + (p-m)\rho_{p-m}^{(n)*}\right]. \tag{A.5}$$

Differentiating the expression (A.5) once again one can obtain after some algebra the equation

$$\ddot{\mathcal{E}}^{(vac)} = 4p^2\mathcal{E}^{(vac)} + 4p(-1)^p\gamma \sum_{n=1}^{\infty} \frac{1}{n}\Phi^{(n)} + 2\mathrm{Re} \sum_{m=1}^{p} m(p-m)^2\left[mF_m + (m+p)G_m\right] \tag{A.6}$$

where

$$\Phi^{(n)} = \mathrm{Im} \sum_{m=1}^{\infty} m^2 \rho_{-m}^{(n)} \left[(p-m)\rho_{p-m}^{(n)*} - (m+p)\rho_{-m-p}^{(n)*}\right]$$

$$F_m = \sum_{n=1}^{\infty} \frac{1}{n}\left[\rho_m^{(n)*}\rho_m^{(n)} - \rho_{-m}^{(n)*}\rho_{-m}^{(n)}\right]$$

$$G_m = \sum_{n=1}^{\infty} \frac{1}{n}\left[\rho_{m+p}^{(n)*}\rho_{m-p}^{(n)} - \rho_{p-m}^{(n)*}\rho_{-m-p}^{(n)}\right].$$

Differentiating the function $\Phi^{(n)}$ over τ and again using equations (2.24), one can verify that the derivative $\mathrm{d}\Psi/\mathrm{d}\tau$ of the combination $\Psi \equiv \sum_{n=1}^{\infty} \frac{1}{n}\left[\Phi^{(n)} + p(-1)^p\gamma S^{(n)}\right]$ can be written in a form analogous to the last sum (from 1 to p) in equation (A.6), but the symbol Re should be replaced by Im. Since $F_m = 1/m$ due to the identity (2.27) and $G_m = 0$ due to (2.28), we have $\mathrm{d}\Psi/\mathrm{d}\tau = 0$. Taking into account the initial conditions $\Phi^{(n)}(0) = S^{(n)}(0) = 0$ one obtains $\Psi(\tau) = 0$. Combining all the terms giving the second derivative of $\mathcal{E}$, one can finally arrive at equation (3.1), where the term $\frac{1}{6}p^2(p^2-1)$ is the value of the sum $2\sum_{m=1}^{p} m(p-m)^2$.

The initial value of the first derivative $\dot{\mathcal{E}}(\tau)$ is determined by the right-hand sides of equations (A.3), (A.4) and (A.5) taken at $\tau = 0$, when $\rho_m^{(n)} = \delta_{mn}$:

$$\dot{\mathcal{E}}(0) = -2p\sigma \sum_{n=1}^{\infty} \sqrt{n(n+p)}\mathrm{Re}\,\langle \hat{b}_n^{\dagger}\hat{b}_{n+p}\rangle - p\sigma \sum_{n=1}^{p-1} \sqrt{n(p-n)}\mathrm{Re}\,\langle \hat{b}_n\hat{b}_{p-n}\rangle.$$

Comparing this formula with (3.3) we arrive at equation (3.4).

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